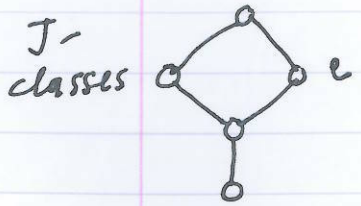
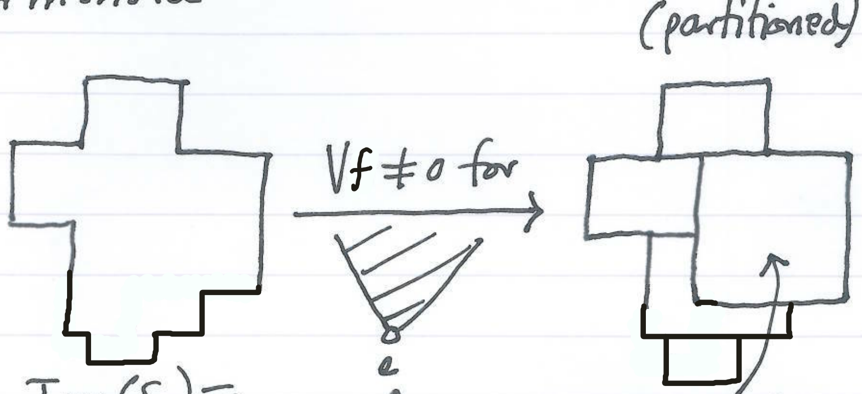
4. Clifford–Munn–Ponizovskiĭ correspondence

S = finite regular monoid



$T = \{e\}$ idempotent representatives



$Irr(S)$ = irreducible S-reps.

apex

$Irr_e(S) = \{V \in Irr(S) : V \text{ has apex } e\}$

we show:

$$\boxed{Irr_e(S)} \underset{U\uparrow S \,\leftarrow\!\shortmid\, U}{\overset{V \mapsto V\downarrow G_e}{\rightleftarrows}} \boxed{Irr(G_e)}$$

bijections.

$\Rightarrow$ CMP correspondence: $Irr(S) \overset{\text{bij.}}{\rightleftarrows} \bigcup_{e \in T} Irr(G_e)$

Prove for $S = I_n$, although (should) easily generalise to any inverse monoid.

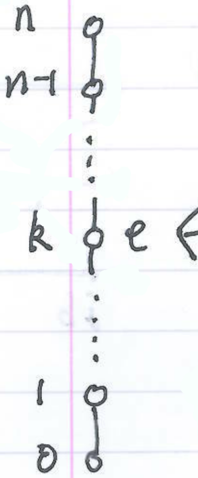
① $\boxed{Irr_e(S)} \xrightarrow{V \mapsto V\downarrow G_e} \boxed{Irr(G_e)}$

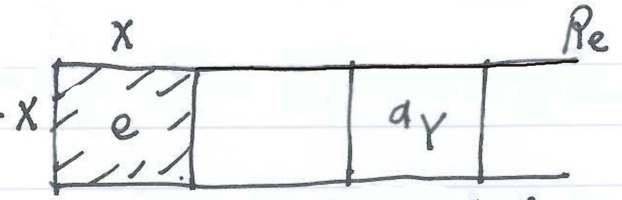
saw: $V\downarrow G_e = Ve$ irreducible G_e-rep. when e = apex of V

($\Rightarrow$ ① is a map)

② $\boxed{\mathrm{Irr}_e(S)} \xleftarrow{\;U\uparrow S \,\leftarrow\!\shortmid\, U\;} \boxed{\mathrm{Irr}(G_e)}$

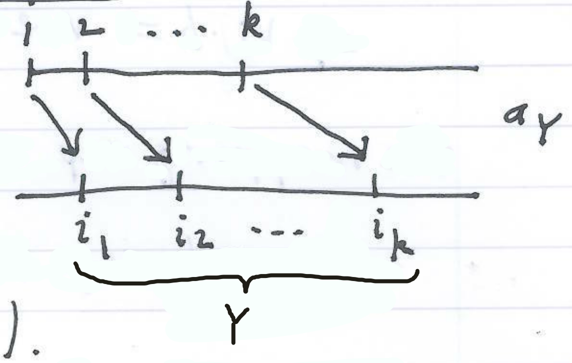
saw: $U\uparrow S$ irreducible.





$X=\{1,\dots,k\}$

$A=\{a_Y\}_{|Y|=k}$



U G_e-rep. ($\cong S_k$-rep).

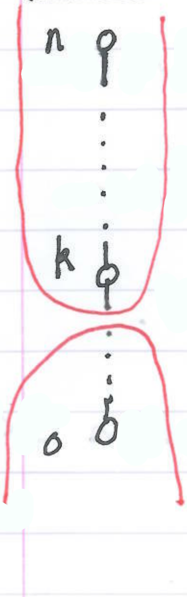
$\Rightarrow U\uparrow S = \bigoplus_{a_Y} U_Y$ with $U_Y=\{u\otimes a_Y : u\in U\}$

(recall: $\mathrm{Ann}(L_e)=0$)

show:

(i). $f = \{1,\dots,l\} \xrightarrow{id} \{1,\dots,l\}$

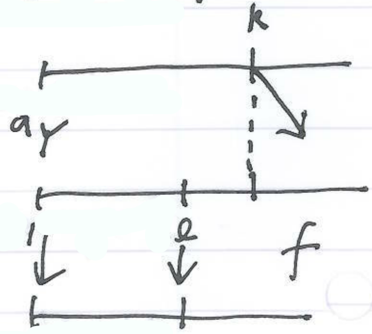
for $l<k$ (ie: $J_f < J_e$)



$(U\uparrow S)f \neq 0$

$(U\uparrow S)f = 0$

$\Rightarrow \mathrm{dom}(a_Y f) \overset{\text{proper}}{\subset} X$ (all Y)



$\Rightarrow a_Y f \notin R_e$

$\Rightarrow (u\otimes a_Y)\cdot f = 0$ (all Y)

$\Rightarrow (U\uparrow S)f = 0$.

(ii). $a_Y e \in R_e \Leftrightarrow \mathrm{dom}(a_Y e) = X \Leftrightarrow Y = X \Leftrightarrow a_Y = e$

i.e: $(u \otimes a_Y) \cdot e \neq 0 \Leftrightarrow u \otimes a_Y = u \otimes e$

$\Rightarrow u \otimes e \mapsto u$ an isom. $(U \uparrow S) e \xrightarrow{\cong} U_X = U$

and

$$\begin{tikzcd} u \otimes e \arrow[r, mapsto, "(-)g"] \arrow[d, mapsto] & (u \otimes e) \cdot g = u \cdot g \otimes e \arrow[d, mapsto] \\ u \arrow[r, mapsto, "(-)g"] & u \cdot g \end{tikzcd} \quad (g \in G_e)$$

commutes $\Rightarrow (U \uparrow S) e \cong U$ as G_e-reps.

<u>conclusion</u>: $-$ $(U \uparrow S) e \neq 0 \Rightarrow e =$ apex of $U \uparrow S$

$\Rightarrow \boxed{\mathrm{Irr}_e(S)} \xleftarrow{U \uparrow S \,\leftarrow\!\!\shortmid\, U} \boxed{\mathrm{Irr}(G_e)}$ is a map

$-$ $(U \uparrow S) \downarrow G_e \cong U \Rightarrow \circlearrowright = id.$

③ $\circlearrowleft$ i.e. $(V \downarrow G_e) \uparrow S$ for V irreducible

S-representation with apex e.

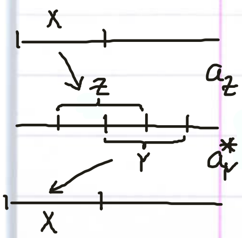
"reconstruct" $(V \downarrow G_e) \uparrow S$ inside V

Consider the $V \cdot (e a_Y)$ subspaces of V

(i). $V \cdot (e a_Y) \cong V \cdot e$ (as spaces) via $v \cdot e \mapsto v \cdot (e a_Y)$

(as $V \cdot e \underset{a_Y^*}{\overset{a_Y}{\rightleftarrows}} V \cdot (e a_Y)$ inverses)

<u>Ex</u>: V an S-rep, f idempotent with $Vf = 0$; then a J-related to $f \Rightarrow Va = 0$.



(ii). $Z \neq Y \Rightarrow V\cdot(ea_Y) \cap V\cdot(ea_Z) = 0$

$$\left(\text{map } V\cdot(ea_Y) \cap V\cdot(ea_Z) \xrightarrow[\cong]{(-)a_Y^*} (V\cdot(ea_Y)\cap V\cdot(ea_Z))\cdot a_Y^*\right.$$

$\subset V\cdot e \cap V\cdot(ea_Z a_Y^*)$; $ea_Z a_Y^*$ J-related to idem. f in a lower J-class $\underset{\text{(Ex.)}}{\overset{e \text{ apex}}{\Longrightarrow}} V\cdot(ea_Z a_Y^*) = 0$).

(iii). S-action on $\bigoplus_{a_Y} V\cdot(ea_Y) \subset V$:

$$a_Y b = \begin{cases} \in R_e \Rightarrow a_Y b = g a_Z, \text{ some } g \in G_e \\ \notin R_e \Rightarrow \operatorname{dom}(a_Y b) \subsetneq X \Rightarrow a_Y b \in J < J_e \end{cases}$$

$$\Rightarrow \quad v\cdot(ea_Y)\cdot b = \begin{cases} (v\cdot g)\cdot(ea_Z) & \text{if } a_Y b \in R_e \\ 0, & \text{else. (by Ex.)} \end{cases}$$

<u>conclusion</u>: — $\bigoplus_{a_Y} V\cdot(ea_Y)$ subrep. of V

with $0 \neq Ve \subset \bigoplus_{a_Y} V\cdot(ea_Y) \underset{\text{irred.}}{\overset{V}{\Longrightarrow}} V = \bigoplus_{a_Y} V\cdot(ea_Y)$

and

$$\begin{array}{ccc} v\cdot(ea_Y) & \xmapsto{(-)b} & \begin{cases} (v\cdot g)\cdot(ea_Z), & a_Y b \in R_e \\ 0 \end{cases} \\ \downarrow & & \downarrow \\ v\cdot e \otimes a_Y & \xmapsto{(-)b} & \begin{cases} v\cdot(eg) \otimes a_Z, & a_Y b \in R_e \\ 0, & \text{else} \end{cases} \end{array}$$

commutes, ie: $V \cong (V\downarrow G_e)\uparrow S$ as S-reps.

$\Rightarrow \circlearrowleft = id.$